

Comment on ‘Critical scalar field collapse in AdS_3 : an analytical approach’

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Abstract

We comment on the derivation of an analytical solution presented in [1], show that it belongs to a family of separable solutions previously constructed in [2], and question its relevance to critical collapse.

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In a recent paper[1], an analytical solution (thereafter referred to as the BST solution), depending on a real parameter p , to the Einstein field equations for a massless scalar field collapsing in AdS_3 was presented, and argued to be relevant to critical collapse in the range $p > 1$. We show here that although the BST solution is indeed a solution of the field equations, its derivation in [1] is incorrect for $p > 1$, and that the BST solution belongs to a larger family of separable solutions to the field equations presented in [2]. We also comment on the relevance of the BST solution to critical scalar field collapse in AdS_3 .

The authors of [1] parameterize the spacetime metric in double-null coordinates by

$$ds^2 = -e^{2\sigma(u,v)} du dv + r(u,v)^2 d\theta^2, \quad (1)$$

and make a self-similar ansatz which leads to the master differential equation for an auxiliary function y depending on the variable $x = -(\alpha/2\sigma) \ln \eta$, where $\eta = u/v$ and α and σ are real constants,

$$\left(\frac{dy}{dx}\right)^2 - \sigma^2 y^2 - \frac{\sigma^2}{2} y^4 = 2E, \quad (2)$$

with E a real integration constant. This can be solved in terms of elliptic functions. Putting $p = \sqrt{1 - 4E/\sigma^2}$, the authors of [1] show that the resulting metric develops an apparent horizon for $p > 1$, i.e. $E < 0$, and argue that the corresponding critical exponent is $\gamma = 1/2$.

As presented in [1], the derivation of the BST solution is formally incorrect for $p > 1$. From Eq. (29) of [1], the constant E is related to another integration constant \tilde{c} by

$$\frac{\tilde{c}}{2} = \frac{\alpha^4}{A^2} \left(\frac{E}{4\sigma^2} \right). \quad (3)$$

The constant \tilde{c} is introduced in the first integral relating two auxiliary functions $f(\eta)$ and $\rho(\eta)$,

$$e^{2\rho(\eta)} = \frac{\tilde{c} f^2(\eta)}{\eta^{\alpha+1}} \quad (4)$$

(Eq. (23) of [1]). This shows clearly that \tilde{c} , and thus also E , is positive definite, so that only the range $p < 1$ is allowed. Notwithstanding this, the metric (39) of [1]

$$ds^2 = -\frac{\alpha^2}{2}(1-p^2) \frac{r^2}{(uv)^{\alpha+1}} du dv + r^2 d\theta^2, \quad r(u,v) = \frac{(uv)^{\alpha/2}}{y}, \quad (5)$$

with $\phi(u, v)$ given by (12) and (20), is indeed a solution of the cosmological Einstein-scalar field equations for all real values of p^2 , including $p^2 > 1$ (and $p^2 < 0$). To check this, it is enough to replace in (1) $-e^{2\sigma}$ by $e^{2\sigma}$, and follow again the steps of the derivation in [1], leading to the solution (5) with $p^2 > 1$.

Now we show that the analytical solution of [1] actually belongs to the larger family of separable solutions¹ to the field equations presented in [2]. It was shown in [2] that the ansatz

$$ds^2 = F^2(T) \left[-dT^2 + dR^2 + G^2(R)d\theta^2 \right], \quad \phi = \phi(T) \quad (6)$$

leads to a solution of the field equations, provided the functions $G(R)$, $F(T)$ and $\phi(T)$ solve the differential equations

$$G'^2 - kG^2 = a, \quad (7)$$

$$\dot{F}^2 - kF^2 = \Lambda F^4 + b^2, \quad (8)$$

$$\dot{\phi} = \sqrt{2b}F^{-1}, \quad (9)$$

where $\dot{} = \partial/\partial T$, $' = \partial/\partial R$, and k , a and the scalar field strength b are real integration constants.

Equation (8) has the same form as Eq. (2). If we choose the special solution of Eq. (7),

$$G(R) = e^{\nu R} \quad (k = \nu^2, a = 0), \quad (10)$$

and transform the coordinates T and R and the function $F(T)$ to double-null coordinates u and v and a function $y(x)$ defined by

$$e^{\nu R} = \frac{\nu}{\sqrt{2b}} (uv)^{\alpha/2}, \quad e^{\nu T} = \left(\frac{u}{v} \right)^{\alpha/2}, \quad (11)$$

$$F(T) = \frac{\sqrt{2b}}{\nu y(x)} \quad (x = -(\nu/\sigma)T), \quad (12)$$

Eq. (8) with $\Lambda = -1$ goes over into Eq. (2) with

$$E = -b^2\sigma^2/\nu^4, \quad (13)$$

and the solution (6) goes over into the $p > 1$ BST solution (5). For completeness, we note that the $p < 1$ BST solution may similarly be obtained from the separable ansatz dual to (6)

$$ds^2 = G^2(R) \left[-dT^2 + dR^2 + F^2(T)d\theta^2 \right], \quad \phi = \phi(R), \quad (14)$$

¹Separable solutions in the context of critical collapse were also considered in the four-dimensional case in [3].

the functions F , G and ϕ solving the differential equations

$$\dot{F}^2 - kF^2 = a, \quad (15)$$

$$G'^2 - kG^2 = -\Lambda G^4 + b^2, \quad (16)$$

$$\phi' = \sqrt{2b}G^{-1}, \quad (17)$$

Finally, we comment on the relevance of the BST solution to critical collapse. The authors of [1] show that the solution (5) with $p^2 > 1$ has an apparent horizon with a mass aspect

$$M_{AH} = r_{AH}^2 \propto \sqrt{p^2 - 1}, \quad (18)$$

while there is no apparent horizon for $p^2 < 1$. They conclude that the critical value of the parameter p is $p = 1$, corresponding to $E = 0$, and the mass critical exponent is $1/2$. As $E = 0$ implies, from Eq. (13), that the scalar field strength b vanishes, this would mean that the critical solution for scalar field collapse is a vacuum solution, and thus is constant curvature. This contradicts the findings of the numerical simulations of [4] and [5], which show a critical regime for a finite critical value of the scalar field amplitude, with a strong spacelike curvature singularity.

Concerning the critical solution for scalar field collapse in AdS_3 , let us recall that Garfinkle [6] constructed a family of continuously self-similar solutions of the $\Lambda = 0$ equations, depending on an integer n , and found that the $n = 4$ solution was a good fit to the numerical data of [4] for critical collapse near the singularity (where the effect of the cosmological constant is negligible). Garfinkle and Gundlach [7] performed the linear perturbation analysis of the Garfinkle solutions, and showed that with suitable boundary conditions the $n = 2$ Garfinkle solution admitted a single growing mode, suggesting that this solution should be the critical one near the singularity. In [8], the Garfinkle solutions were extended to solutions of the full field equations truncated to first order in the cosmological constant Λ , and the zeroth order linear perturbation analysis of [7] was extended to the same order, confirming the results of that analysis. Thus it would seem that, insofar as the boundary conditions used in [7] and [8] are appropriate, the critical solution for scalar field collapse in AdS_3 is an extension of the $n = 2$ Garfinkle solution.

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